APPENDIX A: BROWNIAN TRANSLATION AND ROTATION

1. Brownian Translation

The derivation of the expression for the mean square distance travelled, or mean square angle rotated, by an object undergoing Brownian motion, is surprisingly simple.

For motion in the x-direction, an object of mass m obeys Newton's second law

$$\begin{aligned}\frac{dx}{dt} &= v \\ m\frac{dv}{dt} &= -\beta v + f(t)\end{aligned} \tag{A1}$$

In Eq. (A1), the force on the object due to the impact of molecules is divided into two pieces. One is a steady viscous force $-\beta v$ (calculation of the constant β is discussed in Appendix B), the other is a random force $f(t)$, which is assumed uncorrelated with x. We imagine a collection of identical objects undergoing this motion. Each object suffers a different force, but the average force over the collection is $\overline{f}(t) = 0$. We wish to know the mean square distance $\overline{x^2}$ travelled in time t, for this collection. We assume that the molecules are in thermal equilibrium, with each other and with the objects, so by the equipartition theorem,

$$\frac{1}{2}m\overline{v^2} = \frac{1}{2}kT, \tag{A2}$$

where k is Boltzmann's constant and T is the temperature.

Consider the equations for x^2 and xv, which follow from Eqs. (A1):

$$\begin{aligned}\frac{1}{2}\frac{dx^2}{dt} &= xv \\ m\frac{dxv}{dt} &= mv^2 - \beta xv + xf(t)\end{aligned} \tag{A3}$$

Upon taking the average, over the collection, of Eqs. (A3), one obtains

$$\begin{aligned}\frac{1}{2}\frac{d\overline{x^2}}{dt} &= \overline{xv} \\ m\frac{d\overline{xv}}{dt} &= m\overline{v^2} - \beta\overline{xv}\end{aligned} \tag{A4}$$

since $\overline{xf(t)} = \overline{x}\overline{f}(t) = 0$. Surprisingly, this force, which causes the Brownian motion, appears to plays no role in the subsequent mathematics. However, it does play a role: it is responsible for Eq. (A2), as can be seen by calculating $\overline{vf}(t) \neq 0$ (which we shall not do here, as it is not needed).

One readily sees from the second of Eqs. (A4) that $\overline{xv}$ exponentially decays to a constant, so that the right side vanishes,

$$\overline{xv} = \frac{m}{\beta}\overline{v^2} = \frac{kT}{\beta}, \tag{A5}$$

where Eq. (A2) is utilized in the second step. Putting Eq. (A5) into the first of Eqs. (A4) and integrating, we obtain the desired result:

$$\overline{x^2} = \frac{2kTt}{\beta}. \tag{A6}$$

It is useful to have an expression for the mean distance $\overline{|x|}$. It can be argued that the particle position probability density distribution is well approximated by a gaussian distribution,

$$P(x) = \frac{1}{\sqrt{2\pi\overline{x^2}}}e^{-x^2/2\overline{x^2}}$$

which yields the result

$$\overline{|x|} = 2\int_0^\infty dx x P(x) = \sqrt{\frac{2}{\pi}\overline{x^2}} \approx .80\sqrt{\overline{x^2}}.$$

It is also useful to have an expression for the mean distance $\overline{|r|}$ travelled when there is motion in two dimensions. Then

$$\overline{|r|} = \int_0^\infty r^2 dr \int_0^{2\pi} d\theta \frac{1}{2\pi\overline{x^2}}e^{-r^2/2\overline{x^2}} = \sqrt{\frac{\pi}{2}\overline{x^2}} \approx 1.25\sqrt{\overline{x^2}}.$$

2. Brownian Rotation

For Brownian rotation through angle θ about an axis, for an object of moment of inertia I, the Newtonian equations are

$$\begin{aligned}\frac{d\theta}{dt} &= \omega \\ I\frac{d\omega}{dt} &= -\beta'\omega + \tau(\omega),\end{aligned} \tag{A7}$$

where the equipartition theorem implies $(1/2)I\omega^2 = (1/2)kT$, and the random torque satisfies $\overline{\tau}(\omega) = 0$. Eqs. (A1) and (A7) are precisely analogous, so the result (A6) in this case becomes

$$\overline{\theta^2} = \frac{2kTt}{\beta'}. \tag{A8}$$

APPENDIX B: VISCOUS FORCE AND TORQUE ON A SPHERE AND ELLIPSOID

1. Fluid Flow Equations

The derivation of the expression for the viscous force, felt by an object moving with constant velocity through a fluid, is surprisingly complicated. First one finds the fluid